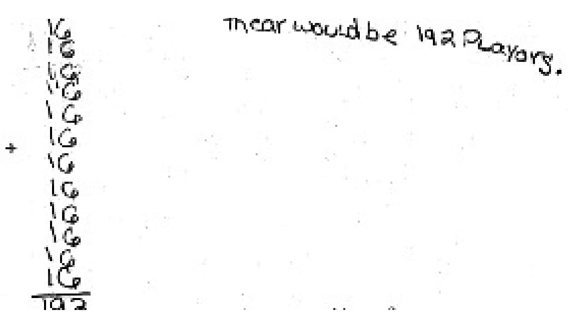
IV. LEARNING TRAJECTORIES AND ADAPTIVE INSTRUCTION MEET THE REALITIES OF PRACTICE[13]

Imagine a 5th-grade teacher is analyzing evidence from student work on a whole number multiplication and division pre-assessment. The pre-assessment consisted of a mix of word problems from a range of contexts and some straight computation problems. She notices one student correctly answered 80% of the problems, but solved the problems using repeated addition or repeated subtraction (Example 1 below). In the past, the teacher might have been pleased that the student had 80% correct. However, she now knows that the use of repeated addition (subtraction) by a 5th-grade student is a long way from that student's attaining an efficient and generalizable multiplicative strategy such as the traditional algorithm (CCSSO/NGA, 2010). She also knows that this student is not ready to successfully engage in the use of new 5th-and 6th-grade concepts like multiplication of decimals (e.g., 2.5 x 0.78), or solving problems involving proportionality, which relies on strong multiplicative reasoning.

Example 1: Use of Repeated Addition (VMP OGAP, 2007)

> There are 16 players on a team in the Smithville Soccer League. How many players are in the league if there are 12 teams?



The teacher observes and records other evidence about the strategies or properties that her students have used to solve the problems (e.g., counting by ones, skip counting, area models, distributive property, the partial products algorithm, and the traditional algorithm); the multiplicative contexts that have caused her students difficulty (e.g., equal groups, multiplicative change, multiplicative comparisons, or measurement); and the types of errors that the students have made (e.g., place value, units, calculation, or equations). She will use this evidence to inform her instruction for the class as a whole, for individual students, and to identify students who could benefit with additional Response to Intervention (RTI) Tier II instruction—a school-wide data-driven system used to identify and support students at academic risk.[14]

This teacher and others like her who have participated in the Vermont Mathematics Partnership Ongoing Assessment Project (VMP OGAP) have used the OGAP Multiplicative Framework (See Appendix B) to analyze student work as briefly described above, to guide their instruction, and engage their students in self-assessment. In addition to administering pre-assessments, they administer formative assessment probes as their unit of instruction progresses. They use the OGAP Framework to identify where along the hypothesized trajectory (non-multiplicative – early additive – transitional – multiplicative) students are at any given time and in any given context, and to identify errors students make.

It is one thing to talk theoretically about learning trajectories and a whole other thing to understand how to transfer the knowledge from learning trajectory research to practice in a way that teachers can embrace it (see Figure 1 below). The latter involves designing tools and resources that serve as ways for classroom teachers to apply the trajectory in their instruction.

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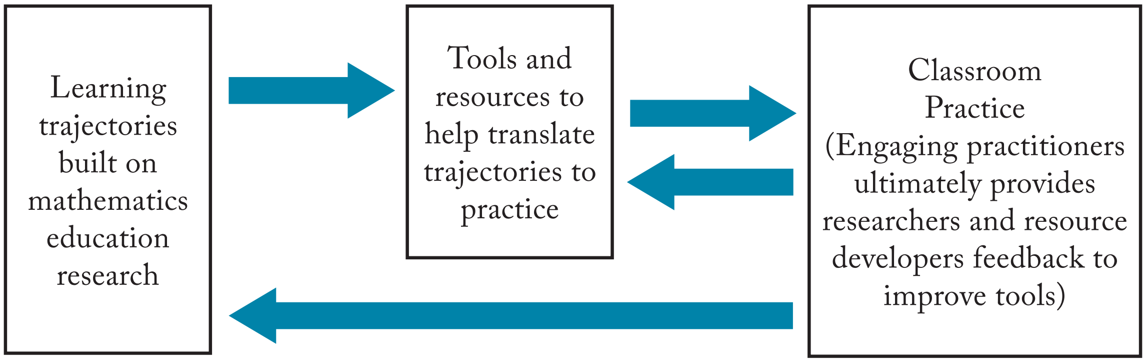
13 Written by Marge Petit, educational consultant focusing on mathematics instruction and assessment. Petit's primary work is supporting the development and implementation of the Vermont Mathematics Partnership Ongoing Assessment Project (OGAP) formative assessment project.

14 There are different levels of intervention. RTI Tier II provides students at academic risk focused instruction in addition to their regular classroom instruction. (http://www.rti4success.org/)

Figure 1. Transfer of Knowledge from Learning Trajectory Research into Classroom Practice



An example of a project that is developing tools and resources that bridge the gap between research and practice is the Vermont Mathematics Partnership Ongoing Assessment Project (OGAP), developed as one aspect of the Vermont Mathematics Partnership (VMP).[15] In 2003, a team of 18 Vermont mathematics educators (classroom teachers, school and district mathematics teacher leaders, an assessment specialist, and a mathematician) were charged with designing tools and resources for teachers to use to gather information about students' learning while they are learning, rather than just after their learning, for the sole purpose of informing instruction. Guided by findings of the NRC's expert panels (Pellegrino, Chudowsky, & Glaser, 2001; Kilpatrick, Swafford, & Findell, 2001), the design team adopted four principles that have guided their work through three studies (VMP OGAP, 2003, 2005, and 2007) involving over 100 teachers and thousands of students: 1) teach and assess for understanding (Kilpatrick, Swafford, & Findell, 2001; 2) use formative assessment intentionally and systematically (Pellegrino, Chudowsky, & Glaser, 2001; 3) build instruction on preexisting knowledge (Bransford, Brown, & Cocking, 2000); and, 4) build assessments on knowledge of how students learn concepts (Pellegrino, Chudowsky, & Glaser, 2001). Incorporating these elements into the tools and resources being developed provided a structure for helping OGAP teachers to engage in adaptive instruction as defined in the introduction to this report.

The fourth principle, build assessments on how students learn concepts, led, over time, to the development of item banks with hundreds of short, focused questions designed to elicit developing understandings, common errors, and preconceptions or misconceptions that may interfere with solving problems or learning new concepts. These questions can be embedded in instruction and used to gather evidence to inform instruction. Importantly, the OGAP design team developed tools and strategies for collecting evidence in student work. One of these tools is the OGAP Frameworks; for multiplication, division, proportionality, and fractions. Teachers use the frameworks to analyze student work and adapt instruction (See, for example, the OGAP Multiplicative Framework in Appendix B). Each OGAP Framework was designed to engage teachers and students in adaptive instruction and learning. Teachers studied the mathematics education research underlying the OGAP Frameworks, and put what they learned into practice. The OGAP Frameworks have three elements: 1) analysis of the structures of problems that influence how students solve them, 2) specification of a trajectory that describes how students develop understanding of concepts over time, and 3) identification of common errors and preconceptions or misconceptions that may interfere with students' understanding new concepts or solving problems.

From a policy perspective, an important finding from the Exploratory OGAP studies and the OGAP scale-up studies in Vermont and Alabama is that teachers reported that knowledge of mathematics education research and ultimately the OGAP Frameworks/trajectories helped them in a number of important ways. They reported that they are better able to understand evidence in student work, use the evidence to inform instruction, strengthen their first-wave instruction, and understand the purpose of the

[15] The Vermont Mathematics Partnership was funded by NSF (EHR-0227057) and the USDOE (S366A020002).

activities in the mathematics programs they use and in other instructional materials (VMP OGAP, 2005, 2007 cited in Petit, Laird, & Marsden, 2010).

The OGAP 2005 and 2007 studies present promising evidence that classroom teachers, when provided with the necessary knowledge, tools, and resources, will readily engage in adaptive instruction. However, other findings from the OGAP studies provide evidence that developing tools and providing the professional development and ongoing support necessary to make adaptive instruction a reality on a large scale will involve a considerable investment and many challenges.

To understand the challenges encountered in implementing adaptive instruction better we return to the teacher who observed a 5th-grade student using repeated addition as the primary strategy to solve multiplication problems. This teacher has made a major, but difficult transition from summative thinking to formative/adaptive thinking. She understands that looking at just the correctness of an answer may provide a "false positive" in regards to a 5th-grade student's multiplicative reasoning. She notices on the OGAP Multiplicative Framework that repeated addition is a beginning stage of development and that 5th-grade students should be using efficient and generalizable strategies like partial products or the traditional algorithm. On a large-scale assessment one cares if the answer is right or wrong. On the other hand, from a formative assessment/adaptive instruction lens, correctness is just one piece of information that is needed. A teacher also needs to know the strategies students are using, where they are on a learning trajectory in regards to where they should be, and the specifics about what errors they are making on which mathematics concepts or skills. This is the information that will help teachers adapt their instruction.

This transition from summative to formative/adaptive instruction was a major challenge for OGAP teachers who were well conditioned to administrating summative assessments ranging from classroom quizzes and tests to state assessments, all of which have very strict administration procedures. In formative assessment/adaptive instruction thinking your sole goal is to gather actionable information to inform instruction and student learning, not to grade or evaluate achievement. That means if the evidence on student work isn't clear—you can ask the student for clarification or ask the student another probing question.

OGAP studies showed that once a teacher became comfortable with looking at student work (e.g., classroom discussions, exit questions, class work, and homework) through this lens, their next question was—"Now that we know, what do we do about it?" As a case in point, one of the best documented fraction misconceptions is the treatment of a fraction as two whole numbers rather than as a quantity unto itself (Behr, Wachsmuth, Post, & Lesh, 1984; VMP OGAP, 2005, 2007; Petit, Laird, & Marsden, 2010; Saxe, Shaughnessy, Shannon, Langer-Osuna, Chinn, & Gearhart, 2007). This error results in students adding numerators and denominators when adding fractions, or comparing fractions by focusing on the numerators or denominators or on the differences between them. Example 2 below from a 5th-grade classroom is particularly troubling, and very informative. In the words of one teacher, "In the past I would have been excited that a beginning 5th-grade student could add fractions using a common denominator. I would have thought my work was done. It never occurred to me to ask the student the value of the sum." (VMP OGAP, 2005). When faced with evidence such as found in Example 2, OGAP teachers made the decision to place a greater instructional emphasis on the magnitude of fractions and the use of number lines, not as individual lessons as they found them in their text materials, but as a daily part of their instruction.

Example 2: Inappropriate Whole Number Reasoning Example

Added sums accurately and then used the magnitude of the denominator or numerator to determine that is closest to 20. (Petit, Laird, & Marsden, 2010)

The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to

A. 20
B. 8
C. $\frac{1}{2}$
D. 1

Explain your answer.

$\frac{1}{12}+\frac{7}{8}=\frac{2}{24}+\frac{21}{24}=\frac{23}{24}$ is closest to 20.

This action is supported by mathematics education research that suggests that number lines can help to build understanding of the magnitude of fractions and build concepts of equivalence (Behr & Post, 1992; Saxe, Shaughnessy, Shannon, Langer-Osama, Chinn, & Gearhart, 2007; VMP OGAP, 2005 and 2007). Research also suggests the importance of focusing on the magnitude of fractions as students begin to operate with fractions (Bezuk & Bieck, 1993, p.127; VMP OGAP, 2005, 2007 cited in Petit, Laird, & Marsden, 2010).

This example has other implications for making adaptive instruction a reality in mathematics classrooms. Resources, like OGAP probes and frameworks, must be developed that are sensitive to the research. Teachers must receive extensive training in mathematics education research on the mathematics concepts that they teach so that they can better understand the evidence in student work (from OGAP-like probes or their mathematics program) and its implications for instruction. They need training and ongoing support to help capitalize on their mathematics program's materials, or supplement them as evidence suggests and help make research-based instructional decisions.

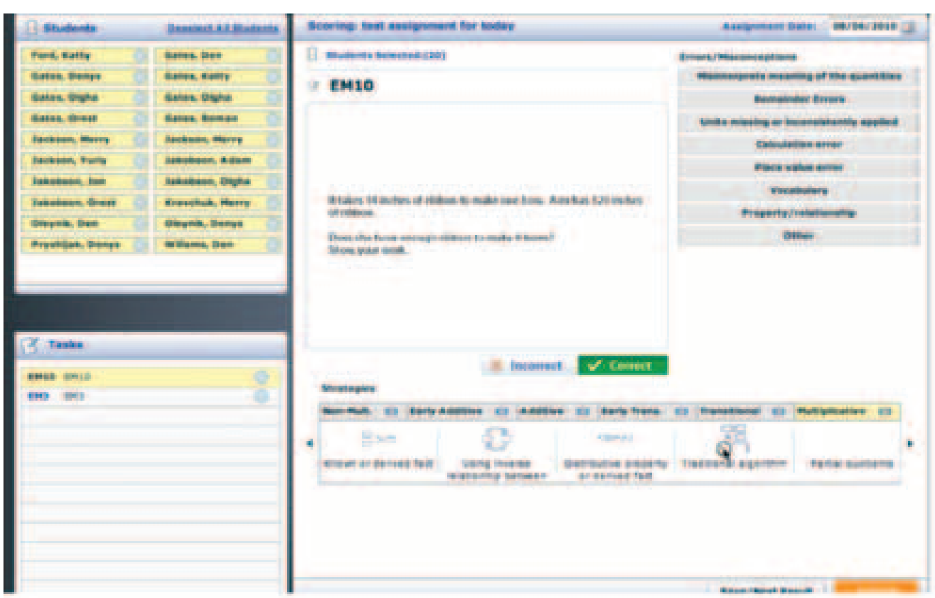
> I realized how valuable a well designed, research-based probe can be in finding evidence of students' understanding. Also, how this awareness of children's thinking helped me decide what they (students) knew versus what I thought they knew. (VMP OGAP, 2005 cited in Petit and Zawojewski, 2010, p. 73)

In addition, while it is true that formative assessment provides teachers the flexibility "to test their interpretations by acting on them and seeing whether or not they get the expected response from the students—and acting again if they don't" (see Section III of this report), OGAP studies show that teachers who understand the evidence in student work from a research perspective are looking for research-based interventions. Drawing on my own experience as a middle school teacher in the early 1990s when I was faced with students adding numerators and denominators (e.g., $\frac{3}{4} + \frac{7}{8} = \frac{10}{12}$), I would re-teach common denominators "louder and slower," never realizing that the problem was students' misunderstanding magnitude or that students did not have a mental model for addition of fractions as suggested in the research.

While there is research on actions to take based on evidence in student work, much more needs to be done if the potential of adaptive instruction is to be realized. Research resources need to be focused not only on validating trajectories as a research exercise, but on providing teachers with research-based instructional intervention choices.

OGAP teachers are now recording on paper a wealth of information on student learning as described earlier in this chapter. To help facilitate this process, OGAP is working closely with CPRE researchers from the University of Pennsylvania and Teachers College, Columbia University, and with the education technology company, Wireless Generation, in developing a technology-based data entry and reporting tool grounded on the OGAP Multiplicative Framework. The tool will be piloted in a small Vermont-based study during the 2010-2011 school year. It is designed to make the item bank easily accessible; it provides a data collection device based on the OGAP Multiplicative Framework linked to item selection (See Figure 2). The tool is designed

Figure 2: Draft Evidence Collection Tool that Uses Touch Screen Technology.



to provide reports that show where on the trajectory (OGAP Framework) each student is at any given time, with any given problem structure, and across time. It also provides results about accuracy and errors, and misconceptions by students and by the class.

Students' performances with respect to learning trajectories, like those in the OGAP Frameworks, do not simply increase monotonically. Rather students move back and forth along the trajectory as they interact with new contexts or more complex numbers until they have fully developed their multiplicative reasoning (VMP OGAP 2005, 2007; Clements, & Sarama, 2009). Development of tools, like the Wireless Generation tool being piloted, will need to account for this movement if they are to represent learning trajectories in a meaningful way.

A very important point here is that OGAP and the Wireless tool being developed is NOT taking the teacher out of the equation as some multiple choice-based diagnostic assessments are purporting to do, to make it easier for a teacher. Rather, the project has recognized the importance of empowering the teacher with knowledge of the research that they use when analyzing student work and making instructional decisions. These are the cornerstones of adaptive instruction. Our hypothesis continues to be that it is the knowledge of the mathematics education research that empowers teachers, not just the data from the results of assessments.

From a policy perspective, to accomplish implementation of adaptive instruction on a large scale our work has shown the importance of capitalizing on existing resources and strategies. In Vermont, this meant working with mathematics teacher leaders who were graduates of a three-year masters program in mathematics (the Vermont Mathematics Initiative). OGAP professional development was provided directly to teacher leaders in two stages. The first stage focused on teacher leader knowledge, and the second phase provided the teacher leaders with support as they worked with other teachers in their district.

Small pilots in Alabama have led to a decision by the Alabama Department of Education to make OGAP a major intervention strategy. Next June AMSTI (Alabama Mathematics and Science Teachers Initiative) leaders from across Alabama will receive OGAP training and support as they begin to engage Alabama teachers state wide. They recognize this is a multi-year effort, but they are setting the stage for it to begin.

Other district and state policies that value the use of formative assessment and adaptive instruction need to be put into place if these strategies are to be used at all by teachers. State standards that favor breadth over depth, or are not built on mathematics education research or districts' use of unrealistic pacing guides linked to quarterly assessments will all serve as formidable barriers to the use of formative assessment and adaptive instruction.

Our work indicates that it is possible to engage teachers in adaptive instruction and to use learning trajectories as described above, but it will take a commitment by policymakers, material developers, mathematics education researchers, and educators at all levels to accomplish the goal.

REFERENCES

ACT (2010). A first look at the common core and college and career readiness. Retrieved from http://www.act.org/research/policymakers/pdf/FirstLook.pdf

Alonzo, A., & Gotwals, A. (Eds.) (in press). *Learning progressions in science.* Rotterdam, The Netherlands: Sense Publishers. Manuscript in preparation.

Barrett, J. E., & Battista, M.T. (in press 2011). A case study of different learning trajectories for length measurement. In J. Confrey, A. Maloney, and K. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education.* Raleigh, North Carolina: Information Age Publishers.

Barrett, J., Clements, D., Cullen, C., McCool, J., Witkowski, C., & Klanderman, D. (2009, April). Children's abstraction of iterative units to measure linear space: A trajectory. Paper presented at the *Annual Meeting of the American Educational Research Association* (AERA). San Diego, California.

Battista, M. T. (2006). Understanding the development of students' thinking about length. *Teaching Children Mathematics, 13*(3), 140-146.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Lester (Ed.), *Second Handbook of Research on Mathematical Teaching and Learning* (pp. 843-908). Reston, VA: National Council of Teachers of Mathematics.

Behr, M. & Post, T. (1992). Teaching rational number and decimal concepts. In T. Post (Ed.), *Teaching mathematics in grades K-8: Research-based methods* (2nd ed.) (pp. 201-248). Boston: Allyn and Bacon.

Behr, M., Wachsmuth, I., Post, T., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Bezuk, N. S., & Bieck, M. (1993). Current research on rational numbers and common fractions: Summary and implications for teachers. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 118-136). New York: Macmillan.

Black, P., & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education: Principles, Policy and Practice, 5*(1), 7-74.

Bransford, J., Brown, A., & Cocking, R. (2000). *How people learn: Brain, mind, experience, and school.* Washington, D.C.: National Academy Press.

CCSSO/NGA. (2010). *Common core state standards for mathematics.* Washington, DC: Council of Chief State School Officers and the National Governors Association Center for Best Practices. Retrieved from http://corestandards.org/

Clements, D.H. (1999). Teaching length measurement: Research challenges. *School Science and Mathematics, 99* (1), 5-11.

Clements, D., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81-89.

Clements, D., & Sarama, J. (2007a). *Building Blocks—SRA Real Math, Grade PreK.* Columbus, OH: SRA/McGraw-Hill.

Clements, D., & Sarama, J. (2007b). Early childhood mathematics learning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 461-555). New York: Information Age Publishing.

REFERENCES

Clements, D. H., & Sarama, J. (2007c). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *Journal for Research in Mathematics Education, 38*, 136-163.

Clements, D., & Sarama, J. (2008). Experimental evaluation of the effects of a research-based preschool mathematics curriculum. *American Educational Research Journal, 45*(2), 443-494.

Clements, D., & Sarama, J. (2009). *Learning and Teaching Early Math: The learning Trajectories Approach.* New York: Routledge.

Clements, D., Swaminathan, S., Hannibal, M., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education, 30(*2), 192-212.

Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119-142.

The College Board (n.d.). Archived AP Data: Exam data from administrations since 1997. Retrieved from http://professionals.collegeboard.com/data-reports-research/ap/archived

Confrey, J. (2008). *A synthesis of the research on rational number reasoning: A learning progressions approach to synthesis*. Paper presented at The 11th International Congress of Mathematics Instruction., Monterrey, Mexico.

Confrey, J., & Maloney, A. P. (in press 2010). Next generation digital classroom assessment based on learning trajectories in mathematics. In C. Dede and Richards, J. (Eds), S*teps towards a digital teaching platform*. New York: Teachers College Press.

Confrey, J., Maloney, A., & Nguyen, K. (in press, 2011). *Learning over time: Learning trajectories in mathematics education.* Charlotte, NC: Information Age Publishers.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). *Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories*. Paper presented at The 33rd Conference of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece.

Corcoran, T., Mosher, F. A., & Rogat, A. (2009). *Learning Progressions in Science: An Evidence-based Approach to Reform* (Research Report #RR-63). Philadelphia: Consortium for Policy Research in Education.

Cross, C., Woods, T., Schweingruber, H. (Eds.) (2009). *Mathematics in early childhood: Learning paths toward excellence and equity.* Committee on Early Childhood Mathematics, Center for Education, National Research Council. Washington, D.C.: National Academy Press.

Dweck, C. S. (1999). *Self-theories: Their role in motivation, personality and development*. Philadelphia: The Psychology Press.

Dweck, C. S. (2002). Messages that motivate: How praise molds students' beliefs, motivation, and performance (in surprising ways). In J. Anderson (Ed.), *Improving academic achievement* (pp. 38-60). New York: Academic Press.

Elliot, A. J., & Murayama, K. (2008). On the measurement of achievement goals: Critique, illustration, and application. *Journal of Educational Psychology, 100*(3), 613-628.

Gelman, R., & Gallistel, C. (1986). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Ginsburg, H. (1983). *The development of mathematical thinking*. New York: Academic Press.

Glasersfeld, E. V. (1995). Sensory experience, abstraction, and teaching. In L. P. Steffe & J. Gale (Eds.), *Constructive Education* (pp. 369-383). Mahwah, NJ: Lawrence Erlbaum Associates.

Greenough, W. T., Black, J. E., & Wallace, C.S. (1987). Experience and brain development. *Children Development*, 58: 539-559.

REFERENCES

Karmiloff-Smith, A. (1992). *Beyond modularity: A developmental perspective on cognitive science*. Cambridge, MA, MIT Press.

Kemple, J. (2010, November). *Children first and student outcomes: 2003-2010*. Paper presented at Children's First retrospective conference, New York City, NY: New York City Reform Retrospective Project.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.

Maloney, A. P., & Confrey, J., (2010, July). *The construction, refinement, and early validation of the equipartitioning learning trajectory.* Proceedings of the 9th International Conference of the Learning Sciences, Chicago.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30* (2), 122-147.

Murata, A., & Fuson, K. C. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using 10. *Journal for Research in Mathematics Education, 37*(5), 421-456.

Murayama, K., & Elliot, A. J. (2009). The joint influence of personal achievement goals and classroom goal structures on achievement-relevant outcomes. *Journal of Educational Psychology, 101*(2), 432-447.

National Mathematics Advisory Panel. *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*, U.S. Department of Education: Washington, DC, 2008.

Nührenbörger, M. (2001). Insights into children's ruler concepts—Grade-2 students' conceptions and knowledge of length measurement and paths of development. In M.V.D. Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology in Mathematics Education, 3*, 447-454. Utrecht, The Netherlands: Freudenthal Institute.

Nunes, T., Light, P., & Mason, J. H. (1993). Tools for thought. *Learning and Instruction, 3*(1), 39-54.

Pellegrino, J., Chudowsky, N., & Glaser, R. (2001). *Knowing what students know: The science and design of educational assessment.* Committee on the Foundations of Assessment, National Research Council. Washington, D.C.: National Academy Press.

Petit, M., Laird, R., & Marsden, E. (2010). *A focus on fractions: Bringing Research to the Classroom.* New York: Routledge.

Petit, M., & Zawojewski, J. (2010). Formative Assessment in the Elementary Classroom. In D. Lambdin & F. Lester (Eds.), *Teaching and learning mathematics: Translating research for elementary teachers.* Reston, VA: National Council of the Teachers of Mathematics.

Piaget, J. (1970). Genetic epistemology (E. Duckworth, translation). New York: Columbia University.

Rampey, B., Dion, G., & Donahue, P. (2009). *The nation's report card: Trends in academic progress in reading and mathematics 2008*. National Assessment of Educational Progress. Retrieved from http://nces.ed.gov/nationsreportcard/pubs/main2008/2009479.asp#section5

Rupp, A. A., Templin, J., & Henson, R. A. (2010). *Diagnostic measurement: Theory, methods and applications*. New York: Guildford.

Sadler, D. R. (1989). Formative assessment and the design of instructional systems. *Instructional Science, 18*, 119-144.

Sarama, J., & Clements, D. H. (2009a). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.

Sarama, J., & Clements, D. H. (2009b, April). *Scaling up successful interventions: Multidisciplinary perspectives*. Paper presented at the American Educational Research Association, San Diego, CA.

REFERENCES

Saxe, G. B., Shaughnessy, M. M., Shannon, A., Langer-Osuna, J. M., Chinn, R., & Gearhart, M. (2007). Learning about fractions as points on a number Line. In W. G. Martin, M. E. Strutchens, & P. C. Elliott (Eds.), *The learning of mathematics: Sixty-ninth yearbook* (pp. 221-237). Reston, VA: National Council of Teachers of Mathematics.

Saxe, G., Taylor, E., McIntosh, C., & Gearhart, M. (2005). Representing fractions with standard notation: A developmental analysis. *Journal for Research in Mathematics Education, 36*(2), 137-157.

Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1997). *A splintered vision: An investigation of U.S. science and mathematics education executive summary*. Washington, D.C: U.S. National Research Center for the Third International Mathematics and Science Study, Michigan State University.

Sherin, B., & Fuson, K. (2005). Multiplication strategies and the appropriation of computational resources. *Journal for Research in Mathematics Education, 36*(4), 347-395.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructive perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M., Saldanha, L., McClintock, E., Karagoz Akar, G., Watanabe, T., & Ozgur Zembat, I. (2010). A developing approach to studying students' learning through their mathematical activity. *Cognition and Instruction, 28*(1), 70-112.

Sophian, C. (2007). *The origins of mathematical knowledge in childhood.* New York: Routledge.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

Steffe, L., & Olive, J. (2009). *Children's fractional knowledge*. New York: Springer.

Tyler, D., & McKenzie, B. (1990).Spatial updating and training effects in the first year of human infancy. *Journal of Experimental Child Psychology, 50*(3), 445–461.

Van den Heuvel-Panhuizen, M. (Ed.) (2008). *Children learn mathematics: A learning-teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school*. Rotterdam, The Netherlands: Sense Publishers.

Vermont Mathematics Partnership's Ongoing Assessment Project (VMP OGAP) (2003). *Unpublished student work samples and data.* Montpelier, Vermont.

Vermont Mathematics Partnership's Ongoing Assessment Project (VMP OGAP) (2005). *Unpublished student work samples and data.* Montpelier, Vermont.

Vermont Mathematics Partnership's Ongoing Assessment Project (VMP OGAP) (2007). *Unpublished student work samples and data.* Montpelier, Vermont.

Vygotsky, L. (1978). *Mind in society.* Cambridge, MA: Harvard University Press.

West, P., Rutstein, D., Mislevy, R., Liu, J., Choi, Y., Levy, R.,...Behrens, J. (2010, August). *A bayesian network approach to modeling learning progressions and task performance.* National Center for Research on Evaluation, Standards, and Student Testing. UCLA: Los Angeles.

Wilson, M. (2005).Constructing measures: An Item Response Modeling approach. Mahwah, NJ: Erlbaum.

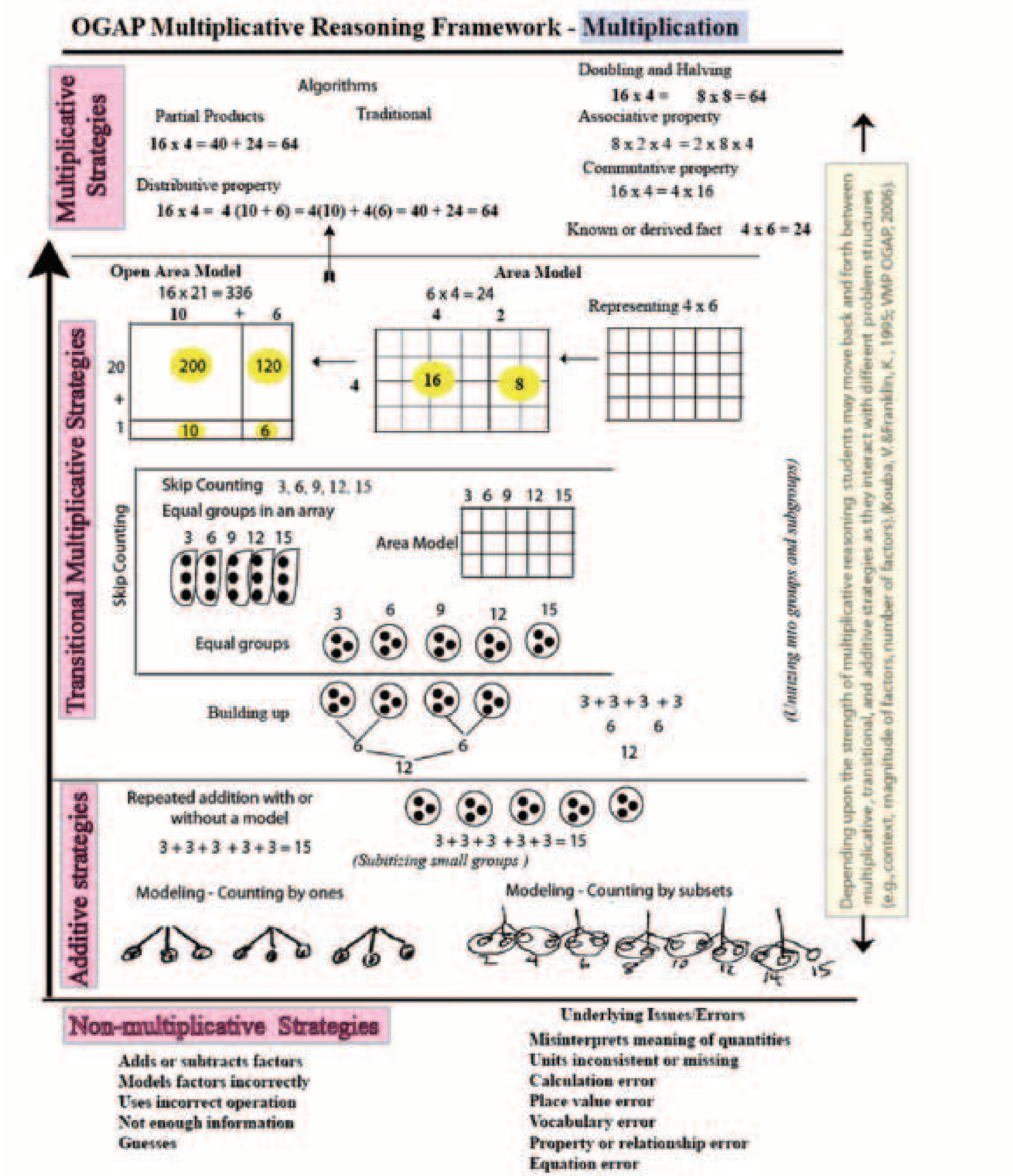
Wiser, M., Smith, C. L., Asbell-Clarke, J., & Doubler, S. (2009, April). *Developing and refining learning progression for matter: The inquiry project, Grades 3-5*. Paper presented at the Annual Meeting of the American Educational Research Association, San Diego.

Wu, H. (2008). *Fractions, decimals and rational numbers.* Retrieved from http://math.berkeley.edu/~wu/NMPfractions4.pdf.

Xu, F., Spelke, E., & Goddard, S. (2005). Number sense in human infants. *Developmental Science, 8*(1), 88-101.

APPENDIX B: OGAP MULTIPLICATIVE REASONING FRAMEWORK—MULTIPLICATION

September 2009



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